NONEXISTENCE OF A SHOCK LAYER
IN GAS DYNAMICS
WITH A NONCONVEX EQUATION OF STATE

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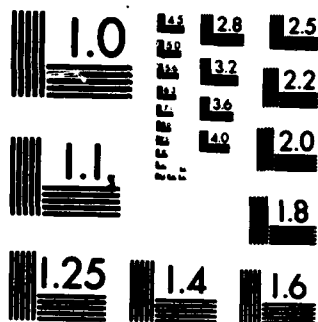
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ABSTRACT

A classical result of Gilbarg states that a simple shock wave solution of

Euler's equations is compressive if and only if a corresponding shock layer

solution of the Navier-Stokes equations exists, assuming, among other things,

that the equation of state is convex. An "entropy condition" appropriate for

weeding out "unphysical" shocks in the nonconvex case has been introduced by

T.-P. Liu. For shocks satisfying his entropy condition, Liu showed that

purely viscous shock layers exist (with zero heat conduction). Dropping the

convexity assumption, but retaining many other reasonable restrictions on the

equation of state, Pego, the author constructs an example of a (large amplitude) shock which

satisfies Liu's entropy condition but for which a shock layer does not exist

if heat conduction dominates viscosity. Pego. We also gives a simple restriction,

weaker than convexity, which does guarantee that shocks which satisfy Liu's

entropy condition always admit shock layers.

AMS (MOS) Subject Classifications: 76N10, 35Q10, 76L05, 35L65, 35B99

Key Words: shocks, shock layer, traveling waves, conservation laws,

admissibility, entropy condition, viscosity, heat conduction

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SIGNIFICANCE AND EXPLANATION

A traditional method of determining which shocks (which are discontinuous solutions in gas dynamics or elasticity) are "physical" is to ask whether the discontinuity can be obtained in the limit of vanishing viscosity and heat conduction. For a constant profile shock, one then seeks corresponding constant profile "shock layer" solutions when viscosity and heat conduction are nonzero. If the equation of state is convex, a simple condition distinguishes those discontinuities which admit shock layers: The shock must be compressive, or entropy should increase across the jump. Convexity does not hold for some materials, however. In the nonconvex case, Liu proposed a revised "entropy condition" to select physical shocks. For small amplitude shocks, the present author has shown that a shock layer always exists if Liu's entropy condition is satisfied. Here we present an example to show that this is not the case for strong shocks, unless the equation of state satisfies some extra restrictions. We also supply some extra restrictions which are shown to be appropriate.

NONEXISTENCE OF A SHOCK LAYER IN GAS DYNAMICS
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§1. Introduction

In 1951, D. Gilbarg [2] showed that for each stationary shock wave
solution of Euler's equations of gas dynamics in one space dimension, there
exists a corresponding smooth solution (called a shock layer or shock profile)
of the Navier-Stokes equations with viscosity and heat conduction, provided
that the thermodynamic equation of state satisfies a short list of
restrictions given by Weyl [11], including a convexity condition. The shock
wave is required to satisfy the Rankine-Hugoniot jump relations, and the
entropy condition, which ensures that entropy increases along particle paths
in Euler's equations. Under Weyl's restrictions, the entropy condition simply
requires shocks to be compressive, i.e., density must increase along particle
paths.

More recently, interest has developed in equations of state which may be
nonconvex ([5],[9],[10]). T.-P. Liu introduced an entropy condition
appropriate for shocks in this situation. He showed that a shock satisfies
his entropy condition if and only if a purely _viscous_ shock layer exists,
i.e., a shock layer for the equations with zero heat conduction [4].

The purpose of this note is to present an example concerning restrictions
on the equation of state under which Liu's entropy condition does select just
those shocks for which shock layers exist for any positive values of viscosity
and heat conduction. We explicitly construct an equation of state such that

for some shock wave which satisfies Liu's entropy condition, no corresponding
shock layer can exist if the heat conduction dominates the viscosity. (A
shock layer does exist if heat conduction is small compared to viscosity.)
The equation of state satisfies all of Weyl's restrictions except the
convexity condition, and satisfies other conditions which have appeared in
recent work on the Riemann problem in gas dynamics ([5],[10]). We can give a
simple restriction weaker than convexity (satisfied by van der Waals gases in
regions of hyperbolicity, for example) under which all shocks satisfying Liu's
entropy condition always admit corresponding shock layers. (See Theorem 1.)

An example such as we describe may be of interest in one dimensional
thermo(visco)elasticity, where for quite general constituitive relations it is
desirable to determine the smoothing effect of viscous and thermal dissipation
(e.g. see [1]). Nonconvex equations of state also occur for materials
exhibiting phase transitions. In this paper, however, we require that Euler's
equations remain hyperbolic in the region of interest. Effects other than
viscosity and heat conduction must be considered to determine "structure" in
phase transition zones; see [8] and [9].

The Navier-Stokes equations in one space dimension, written in Lagrangian
form, are

(1.1)
$$\tau_t - u_h = 0$$
$$u_t + p_h = (\mu u_h/\tau)_h$$
$$\mathcal{E}_t + (pu)_h = (\mu u u_h/\tau)_h + (\lambda \theta_h/\tau)_h$$

Here h denotes the Lagrangian mass coordinate, t is time, τ is specific
volume, u is velocity, p is pressure, θ is temperature, $\mathcal{E}$ is energy desity per
unit mass, and μ and λ are the coefficients of viscosity and heat conduction,
respectively. $\mathcal{E} = e + u^2/2$, where e is the internal energy density per unit

mass. We assume that τ and θ determine the thermodynamic state of the material, and that e, p, and the entropy S are given by sufficiently smooth equations of state $e = e(\tau,\theta)$, $p = p(\tau,\theta)$, $S = S(\tau,\theta)$. These functions are related through the Gibbs relation

$$(1.2) \qquad\qquad \theta \; dS \; = \; de \; + \; p \; d\tau$$

The coefficients μ and λ may also depend smoothly on τ and θ. The quantities μ, λ, τ, θ, and p are positive. For convenience, we shall denote the triple $(\tau, u, \&)$ by U, and the thermodynamic state by Z, specified by the pair (τ,θ) (or by (τ,p), see assumption (1.10)).

Euler's equations are obtained from (1.1) by setting $\mu = \lambda = 0$. A shock wave traveling at speed s is a weak solution of these equations of the form

$$(1.3) \qquad\qquad U(h,t) \; = \; \begin{cases} U_- & \text{for } h < st \\ U_+ & \text{for } h > st \end{cases}$$

The two end states must satisfy the Rankine-Hugoniot relations

$$
\begin{aligned}
0 \; &= \; -s(\tau_+ - \tau_-) \; - \; (u_+ - u_-) \\
(1.4) \qquad 0 \; &= \; -s(u_+ - u_-) \; + \; (p_+ - p_-) \\
0 \; &= \; -s(\&_+ - \&_-) \; + \; (p_+ u_+ - p_- u_-)
\end{aligned}
$$

We call any solution (1.3) satisfying (1.4) a "simple jump solution", or "jump". It is customary to reserve the term "shock" for jumps which satisfy the entropy condition.

What determines a jump? The speed s may be determined up to sign from any suitable pair of thermodynamic states Z_+ and Z_- by the relation

$$(1.5) \qquad\qquad 0 \; = \; s^2(\tau_+ - \tau_-) \; + \; (p_+ - p_-)$$

In our analysis it suffices to consider only back-facing shocks, for which $s < 0$. (Particle paths in (1.1) are vertical.) Now $u_+ - u_-$ is

determined. Imposing the third equation, we recall: The states Z_+ and Z_- determine a simple jump solution with $s < 0$ if and only if the __Hugoniot function__ vanishes:

$$H(Z_+,Z_-) \equiv e_+ - e_- + \tfrac{1}{2}(p_+ + p_-)(\tau_+ - \tau_-) = 0$$

Fixing Z_0, the set of states Z satisfying $H(Z,Z_0) = 0$ is typically a curve, called the __Hugoniot curve__ with center Z_0 in the (τ,p) plane.

A __shock layer__ for the jump (1.3) is a traveling wave solution $U(h-st)$ of (1.1) with $U(\xi) \to U_\pm$ as $\xi \to \pm\infty$. Plugging in and integrating once, $U(\xi)$ must satisfy

$$0 = -s(\tau - \tau_\pm) - (u - u_\pm)$$

$$\mu u_\xi/\tau = -s(u - u_\pm) + (p - p_\pm)$$

$$\mu u u_\xi/\tau + \lambda\theta_\xi/\tau = -s(\mathcal{E} - \mathcal{E}_\pm) + pu - p_\pm u_\pm$$

The first equation determines $u(\xi)$ from $\tau(\xi)$, so we obtain

(1.6)
$$\lambda\theta_\xi/\tau = -s(\mathcal{E} - \mathcal{E}_\pm + (\tau - \tau_\pm)(p_\pm + su)) \equiv L(\tau,\theta)$$

$$\mu\tau_\xi/\tau = \frac{1}{-s}(s^2(\tau - \tau_\pm) + p - p_\pm) \equiv M(\tau,\theta)$$

These equations yield an autonomous system of ODEs in the (τ,θ) plane. It is clear that Z_+ and Z_- are rest points for this system. A shock layer exists for the jump (1.2) if and only if the system (1.5) admits a trajectory $Z(\xi)$ connecting Z_- to Z_+, i.e., $Z(\xi) \to Z_\pm$ as $\xi \to \pm\infty$.

We remark that equations (1.6) are identical to those used by Gilbarg [2] for the stationary shock layer in Eulerian form, if we introduce an eulerian space variable $x = \int^\xi \tau(\eta)d\eta$ and recognize that $-s$ is the mass flux $\rho u = const$, where $\rho = 1/\tau$ is the density.

In order to motivate Liu's entropy condition for the jump (1.3) and examine the implications of nonconvexity, we briefly consider the simple

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example of <u>isothermal</u> gas dynamics in Lagrangian form. The equations are

$$\tau_t \; - \; u_h \quad = \quad 0$$

$$u_t \; + \; p(\tau)_h \quad = \quad (\mu u_h/\tau)_h$$

A simple jump with speed s connecting (τ_-, u_-) and (τ_+, u_+) for $\mu = 0$ satisfies

$$0 \; = \; s^2(\tau_+ - \tau_-) \; + \; p(\tau_+) \; - \; p(\tau_-)$$

A shock profile $(\tau, u)(\xi)$, with $\xi = h-st$, must satisfy (eliminating u)

$$(1.7) \qquad \mu\tau_\xi/\tau \; = \; \frac{1}{-s}(s^2(\tau - \tau_\pm) \; + \; p(\tau) - p(\tau_\pm)) \; \equiv \; M(\tau)$$

Assuming that $p(\tau)$ is <u>decreasing</u>, but possibly nonconvex, any two states
τ_+, τ_- determine a jump with speed s < 0. A shock layer exists for that jump
if and only if: sgn $M(\tau)$ = sgn $(\tau_+ - \tau_-)$ for τ between τ_+ and τ_- .
This is the "entropy condition" which selects those jumps which admit a shock
layer for these equations. The situation is pictured in Fig. 1, where the
phase portrait of (1.7) is indicated on the τ axis of the graph of $p(\tau)$.
Observe that since p is not convex, rarefaction shocks can exist admitting
shock layers, e.g. $\tau_- = \tau_2 < \tau_+ = \tau_3$. Also, a compressive jump need not
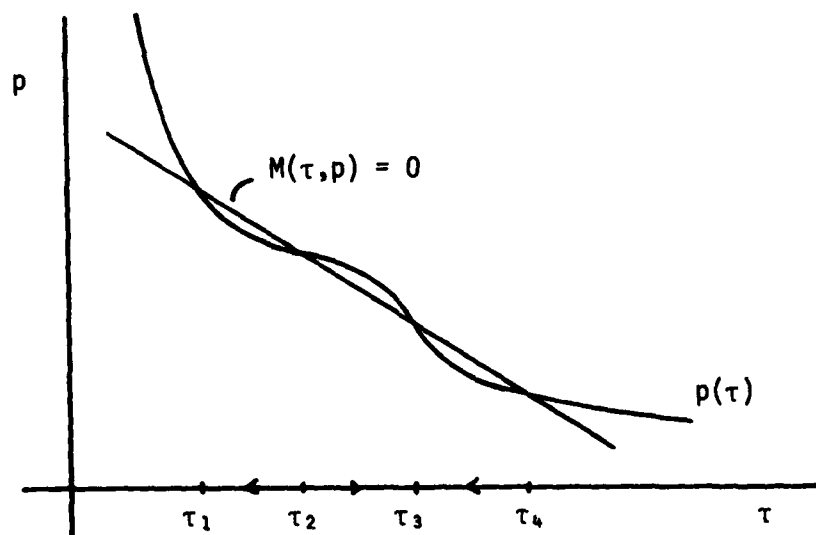admit a shock layer, e.g. $\tau_- = \tau_4 > \tau_+ = \tau_1$.

Figure 1. Shock profiles in isothermal gas dynamics

For the nonisothermal Euler's equations, we now state Liu's strict entropy condition for a jump (1.3) determined by Z_+, Z_- with $s < 0$: the condition is that

$$\text{sgn } M(Z) = \text{sgn}(\tau_+ - \tau_-) \quad \text{for all } Z = (\tau,p)$$

$s(E)$ <u>between</u> Z_+ and Z_- on the Hugoniot curve

with center Z_+ (or with center Z_-).

With $s < 0$, this condition simply means that the line $M(\tau,p) = s^2(\tau - \tau_\pm) + p - p_\pm = 0$ should lie above (below) the Hugoniot curve between Z_+ and Z_- if $\tau_- > \tau_+$ ($\tau_- < \tau_+$). The restrictions we shall impose on the equation of state imply that with some choice of center, the Hugoniot curve is regular between Z_+ and Z_-, so the condition $s(E)$ is well defined.

The restrictions we place on the equation of state are as follows:

$$(1.8) \qquad e_\theta(\tau,\theta) > 0$$

This means that heat conduction is a dissipative effect in (1.1) for $\lambda > 0$.

$$(1.9) \qquad p_\tau(\tau,S) \equiv dp/d\tau \Big|_{S \text{ const}} < 0$$

Euler's equations are strictly hyperbolic just when this holds.

$$(1.10) \qquad p_\theta(\tau,\theta) > 0$$

Thus the map $(\tau,\theta) \to (\tau,p(\tau,\theta))$ is one to one on the domain of states Ω_θ. The image of this map we designate Ω_p. We assume

$$(1.11) \qquad \Omega_p \text{ is convex.}$$

These restrictions above are the same as those imposed by Gilbarg [2], following Weyl [11], except that we omit the convexity assumption

$$d^2p/d\tau^2 \Big|_{S \text{ const}} > 0 .$$

Under these restrictions, the Hugoniot curve is described by the following. The proof is deferred to §2.

<u>Proposition 1.</u> Fix $Z_0 = (\tau_0,p_0)$ in Ω_p. Then in the region $\tau > \tau_0$, the set $\{ Z \in \Omega_p | H(Z,Z_0) = 0 \}$ consists solely of a monotonically decreasing curve

$p = h_0(\tau)$ passing through Z_0.

We now describe our positive results concerning existence and uniqueness of the shock layer for any given μ, λ, positive functions of τ and θ.

__Theorem 1.__ Assume (1.8)-(1.11). Also assume

(1.12) $\quad\quad e_\tau(\tau,\theta) > 0 \quad$ in Ω_θ

Suppose Z_+, Z_- in Ω_θ satisfy $H(Z_+,Z_-) = 0$ and (1.13) below. Then the simple jump solution determined by Z_+, Z_- and $s < 0$ from (1.5) admits a unique shock layer if and only if the strict entropy condition $s(E)$ is satisfied.

In the absence of (1.12), __compressive__ shocks ($\tau_- > \tau_+$ if $s < 0$) always admit a shock layer:

__Theorem 2.__ Assume (1.8)-(1.11). Suppose Z_+, Z_- in Ω_θ satisfy $H(Z_+,Z_-) = 0$ and (1.13) below, and suppose $\tau_- > \tau_+$. Then the conclusion of Theorem 1 holds.

__Weak__ shocks ($|U_+ - U_-|$ small) always admit a shock layer, under the most basic of restrictions:

__Theorem 3.__ Assume only (1.8),(1.9). Fix Z_0 in Ω_θ. Then if Z_+ and Z_- are sufficiently close to Z_0 and satisfy $H(Z_+,Z_-) = 0$, the simple junp solution determined by Z_+, Z_- with $s < 0$ admits a shock layer (unique in a neighborhood of Z_0) if and only if the strict entropy condition $s(E)$ is satisfied.

Theorem 3 is proved in [7] as a consequence of a general theorem on singular viscosity matrices for systems of conservation laws. The proof of Theorems 1 and 2 is extremely similar to that of Gilbarg [2], and will be presented briefly in §2. The extra condition imposed is a technical "domain" condition,

(1.13) $\quad\quad$ The line segment $[\tau_+,\tau_-] \times \{\theta_-\}$ lies in Ω_θ ,

needed so certain constructions are valid. (A similar assumption is made

tacitly by Gilbarg.)

Our main result is a "couterexample":

Theorem 4. There exists an equation of state, defined in a domain Ω_θ, satisfying (1.8)-(1.11), and a pair of states Z_+, Z_- which satisfy $H(Z_+, Z_-) = 0$ and (1.13), such that the jump solution determined by Z_+, Z_- and $s < 0$ satisfies the strict entropy condition $s(E)$, but

- if λ/μ is sufficiently large, no corresponding shock layer exists,
- if λ/μ is sufficiently small, a unique shock layer exists.

From Theorem 2, this shock must be a rarefaction shock, $\tau_- < \tau_+$. We stress, however, that Liu's result shows that a purely viscous shock layer (with $\lambda = 0$) exists, so entropy does increase across this shock, i.e., $S(Z_-) < S(Z_+)$.

This equation of state must violate (1.12) and the convexity condition. However, the following conditions can be satisfied:

$$(1.14) \qquad e_\tau(\tau, p) > 0$$

$$(1.15) \qquad p_\tau(\tau, \theta) < 0$$

Condition (1.14) (along with (1.8)-(1.10), basically) was imposed by Liu [5] to guarantee that the Riemann problem for gas dynamics has a unique solution globally. Condition (1.15), along with (1.8), implies (1.9) and implies that constant states for the Navier-Stokes equations are linearly stable (see [6], [7]).

Finally, we briefly remark that although we have presented this example in the context of gas dynamics, it is valid in the context of 1D thermovisco-elasticity as well. One may redefine the equation of state outside a compact domain so that the restrictions imposed by Dafermos [1] are satisfied. The key step is to appropriately specify the function $p_\theta(\tau, \theta)$ globally (e.g. see section 3).

§2. Existence and uniqueness of the shock layer

We begin this section by gathering several useful thermodynamic calculations, then we proceed to prove Theorems 1 and 2. Take τ and θ as independent variables. Using (1.2), and equating mixed partials of $S(\tau,\theta)$, we obtain the standard identities

$$(2.1) \qquad \theta \, S_\theta(\tau,\theta) \;=\; e_\theta(\tau,\theta)$$

$$(2.2) \qquad \theta \, S_\tau(\tau,\theta) \;=\; e_\tau(\tau,\theta) + p \;=\; \theta p_\theta(\tau,\theta)$$

Considering τ and p independent, we get

$$(2.3) \qquad \theta \, S_p(\tau,p) \;=\; e_p(\tau,p) \;=\; e_\theta(\tau,\theta)\,\theta_p(\tau,p)$$

$$(2.4) \qquad \theta \, S_\tau(t,p) \;=\; e_\tau(\tau,p) + p$$

and also

$$(2.5) \qquad p_\tau(\tau,S) \;=\; -S_\tau(\tau,p) \,/\, S_p(\tau,p)$$

The assumptions (1.8)-(1.10) imply that the derivatives

$S_\theta(\tau,\theta)$, $S_\tau(\tau,\theta)$, $S_p(\tau,p)$ and $S_\tau(\tau,p)$ are all positive.

The following identity helps relate the entropy condition $s(E)$ to the ODEs (1.6).

__Proposition 2.__ Fix Z_+ and Z_- with $H(Z_+,Z_-) = 0$. Let $s < 0$ satisfy (1.5). Now $L(\tau,\theta)$ and $M(\tau,\theta)$ in (1.6) are determined, and for any $Z = (\tau,\theta)$ we have

$$(2.6) \qquad H(Z,Z_\pm) \;=\; \frac{1}{-s}L(Z) \;+\; \tfrac{1}{2}(\tau - \tau_\pm)(-sM)(Z)$$

It follows that

$$H(Z,Z_+) \;-\; H(Z,Z_-) \;=\; \tfrac{1}{2}(\tau_- - \tau_+)(-sM)(Z)$$

__Proof.__ $\quad H(Z,Z_\pm) + L(Z)/s \;=\; \tfrac{1}{2}(u^2 - u_\pm^2) + (\tau - \tau_\pm)(\tfrac{1}{2}p + p_\pm) - p_\pm + su)$

$\qquad\qquad = (\tau - \tau_\pm)(\tfrac{1}{2}s(u + u_\pm) + \tfrac{1}{2}(p - p_\pm) - su) \;=\; \tfrac{1}{2}(\tau - \tau_\pm)(-sM)(Z)$

$\qquad\qquad$ (Recall that $u - u_\pm = -s(\tau - \tau_\pm)$.)

__Proof of Proposition 1.__ With Z_0 fixed, $H(Z,Z_0)$ is a function of $Z = (\tau,p)$. Using (2.2), $H_\tau(\tau,p) = \theta S_\tau(\tau,p) + \tfrac{1}{2}(p_0 - p) > 0$ if $p < p_0$, and

$$H_p(\tau,p) = e_p(\tau,p) + \tfrac{1}{2}(\tau - \tau_0) > 0 \quad \text{if } \tau > \tau_0 .$$

It is convenient now to calculate

(2.7) $\qquad \dfrac{1}{-s}L_\theta(\tau,\theta) = e_\theta(\tau,\theta) \qquad \dfrac{1}{-s}L_\tau(\tau,\theta) = \theta p_\theta(\tau,\theta) + sM(\tau,\theta)$

(2.8) $\qquad -sM_\theta(\tau,\theta) = p_\theta(\tau,\theta) \qquad -sM_\tau(\tau,\theta) = s^2 + p_\tau(\tau,\theta)$

To get L_τ , use Proposition 2 and (2.2).

Proof of Theorem 2. Let Z_+, Z_- in Ω_θ satisfy $H(Z_+,Z_-) = 0$, (1.13), and

$\tau_- > \tau_+$, and let $s < 0$ satisfy (1.5). Suppose first that the entropy

condition s(E) is satisfied. Our proof, like Gilbarg's, is based on an

analysis of the curves $(L = 0)$ and $(M = 0)$ which pass through Z_+ and Z_-.

Since L_θ and M_θ are positive, θ is a given function of τ along each of these

curves, $\theta = \ell(\tau)$, $\theta = m(\tau)$ respectively. We claim that $\ell(\tau)$ and $m(\tau)$ are

defined for $\tau_+ < \tau < \tau_-$ and satisfy

$\qquad$ i) $\qquad \ell(\tau) < m(\tau) \qquad$ for $\tau_+ < \tau < \tau_-$

$\qquad$ ii) $\qquad \ell'(\tau) < 0 \qquad$ for $\tau_+ < \tau < \tau_-$

Indeed, the set $(M = 0)$ in Ω_p is a straight line segment, by (1.11). Pulling

back to the (τ,θ) plane, $m(\tau)$ is defined as described. Using Proposition 2,

the entropy condition s(E) implies that between Z_+ and Z_- on the Hugoniot

curve with center Z_+, $L(Z) > 0 > M(Z)$. Property i) follows. Property ii)

follows from i) using (2.7), since $M < 0$ on the curve $(L = 0)$. The "domain"

condition (1.13), with (1.11), now implies $\ell(\tau)$ is defined as described.

The existence of the shock layer follows from i), ii) (see Fig. 2).

The region $R = \{(\tau,\theta) | \tau_+ < \tau < \tau_-$ and $\ell(\tau) < \theta < m(\tau) \}$ is negatively

invariant under the flow induced by (1.6). Also, any trajectory $Z(\xi)$ which

starts in R must be monotone (θ increasing, τ decreasing) and tend to Z_- as

$\xi \to -\infty$.

Consider a vertical line segment crossing R,

$$\{(\tau,\theta) | \tau = \tau_0 , \ell(\tau) < \theta < m(\tau) \} , \quad \tau_+ < \tau_0 < \tau_- .$$
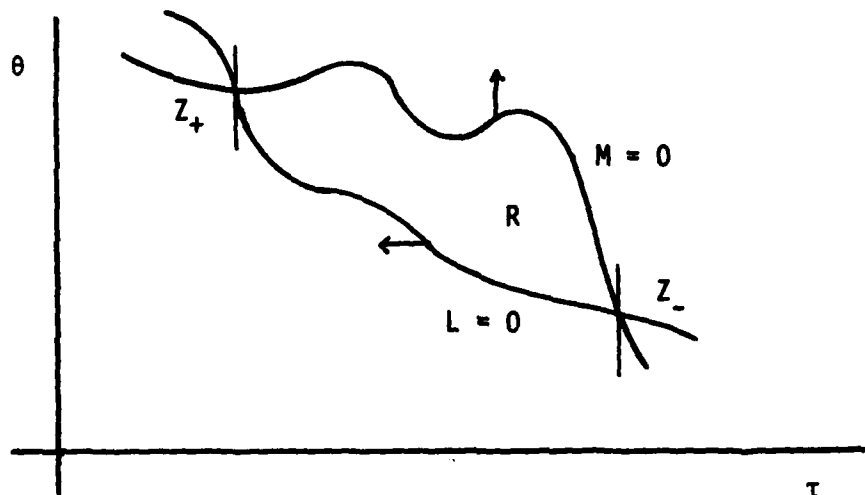
Figure 2. Phase portrait of (1.6) for a compressive shock

A point on this segment may belong to one of three disjoint classes: the forward trajectory of (1.6) starting at the point may

        a)   exit the region R on the curve (L = 0)

        b)   exit the region R on the curve (M = 0)

        c)   not exit the region R .

In the last case, the trajectory must tend toward Z_+ by monotonicity. The first two classes are open (continuity) and nonempty (endpoints). The third class is therefore also nonempty by connectedness, so some shock layer exists.

We turn to demonstrate the uniqueness of the shock layer. First we claim that any trajectory of (1.6) joining Z_+ and Z_- must lie entirely in the region R. Indeed, no trajectory can approach Z_+ within either of the regions (M < 0 and L < 0) or (M > 0 and L > 0). Also, the region

$$R_1 = \{(\tau,\theta) \in \Omega_\theta | \ \tau < \tau_+ \ \text{and} \ L > 0 \ \underline{\text{or}} \ M > 0 \}$$

is negatively invariant. So our claim holds.

The characteristic equation for (1.6) at the critical point Z_+ always has one positive and one nonpositive root:

$$(2.9) \qquad 0 = \begin{vmatrix} M_\tau/\mu - \kappa & M_\theta/\mu \\ L_\tau/\lambda & L_\theta/\lambda - \kappa \end{vmatrix}$$

$$= \kappa^2 - (M_\tau/\mu + L_\theta/\lambda)\kappa + (\ell'(\tau_+) - m'(\tau_+))M_\theta L_\theta/\mu\lambda$$

The constant term is nonpositive. If it is zero, then $M_\tau > 0$. In general, then, Z_+ is a saddle point, and the uniqueness of the trajectory approaching within R is easily established. In the degenerate case, one may construct a one dimensional center-stable manifold in a neighborhood of Z_+, locally invariant under (1.6). A trajectory starting in this neighborhood which does not lie in the given center-stable manifold must eventually (for ξ large) leave any sufficiently small neighborhood of Z_+ (see Kelley [3]). Therefore, any two trajectories approaching Z_+ within R must lie on the same curve.

We have established the existence and uniqueness of the shock layer when $\tau_- > \tau_+$ and the entropy condition s(E) is satisfied. If $\tau_- > \tau_+$ but the condition s(E) fails to hold, arguments similar to those above show that no trajectory of (1.6) can connect Z_- to Z_+. Theorem 2 (and Theorem 1 for compressive shocks) follows.

For a rarefaction shock Z_+, Z_- with $\tau_- < \tau_+$, s < 0 , satisfying the entropy condition s(E), the situation is as follows: L(Z) < 0 < M(Z) for Z between Z_+ and Z_- on the Hugoniot curve with center Z_-. Hence $m(\tau) < \ell(\tau)$ for $\tau_- < \tau < \tau_+$. If $\underline{\ell'(t) < 0}$, then the region

$$R = \{(\tau,\theta) | \ \tau_- < \tau < \tau_+ \ \text{and} \ m(\tau) < \theta < \ell(\tau) \}$$

is negatively invariant and the existence and uniqueness of a trajectory connecting Z_- to Z_+ is established as in the compressive case above. However, it no longer folows from (2.7) that $\ell'(\tau) < 0$ if $\ell(\tau) > m(\tau)$. But we may calculate, using (2.2),

$$-\frac{1}{s}L_\tau(\tau,\theta) = e_\tau(\tau,\theta) + p(\tau,\theta) + sM(\tau,\theta)$$

$$= e_\tau(\tau,\theta) + p(\tau,m(\tau))$$

So the condition $e_\tau(\tau,\theta) > 0$ ensures that $\ell'(\tau) < 0$ for all τ, and

Theorem 1 follows.

We conclude this section by remarking that it is easy to use Propositon 2, (2.7) and (2.8) to show that a simple jump solution admits a purely viscous shock layer ($\lambda = 0$) if and only if the entropy condition $s(E)$ is satisfied. This recovers the result of Liu [4], under the assumptions (1.8)-(1.10), which are rather different than those Liu imposed (see [5]).

§3. A shock without a shock layer

In this section we construct explicitly an equation of state satisfying assumptions (1.8)-(1.10), and (1.14), (1.15), defined in a suitable region of the (τ,θ) plane, but violating the restriction $e_\tau(\tau,\theta) > 0$, such that a (large amplitude) rarefaction shock exists, satisfying the entropy condition $s(E)$, for which **no** shock layer exists if λ/μ is sufficiently large. The shock layer **does** exist if λ/μ is small, and we stress that entropy does increase across this shock, i.e., $S(Z_-) < S(Z_+)$ with $s < 0$.

Our object is to construct Fig. 3, representing the phase portrait for (1.6). The idea is that if λ/μ is large, the vector field $(\tau M/\mu, \tau L/\lambda)$ for (1.6) is nearly horizontal, and a trajectory leaving Z_0 must hit the "hump" in the curve $(M = 0)$. Then it is easy to show that no trajectory connects Z_- to Z_+. But using (2.6) and (2.7), one may easily check that the entropy condition $s(E)$ is satisfied for the jump determined by Z_+, Z_-, and $s < 0$.

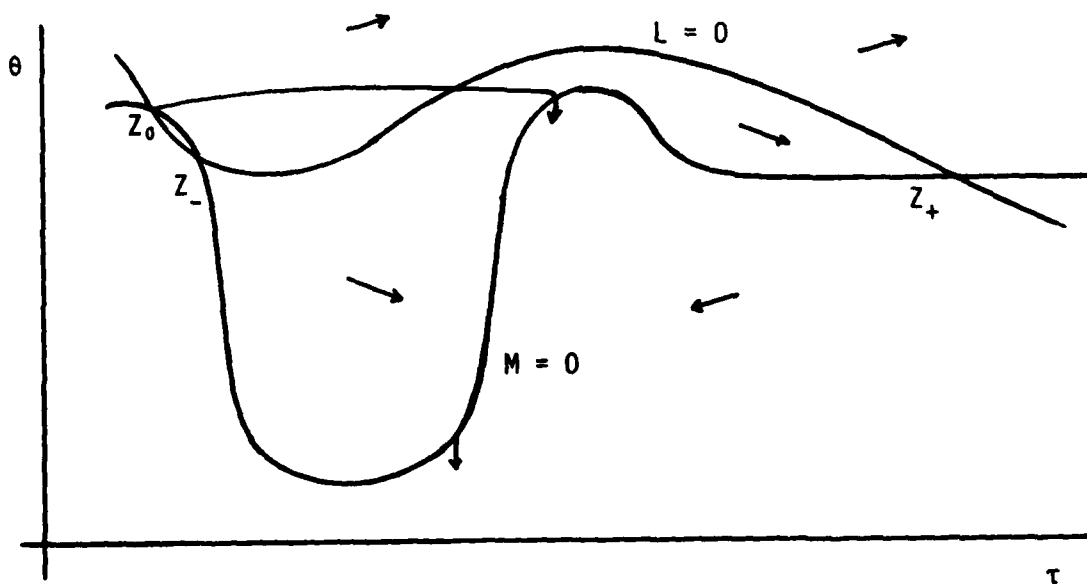

Figure 3. A shock without a profile

Our procedure is to specify selected data (in particular the curve $\theta = m(\tau)$ on which $M = 0$), using thermodynamic identities to consistently specify the equation of state. Fix $Z_0 = (\tau_0, \theta_0)$ in the first quadrant. We shall specify the curve ($M = 0$) as $\theta = m(\tau)$, where $m(\tau)$ satisfies conditions (m1)-(m3) below (so its graph appears as in Fig. 3). Later, we will specify $p_0 = p(\tau_0, \theta_0)$ and s. Then we will know p along the curve ($M = 0$):

(3.1) $\qquad p_m(\tau) \equiv p(\tau, m(\tau)) = p_0 - s^2(\tau - \tau_0)$

For all (τ, θ) , we specify that

(3.2) $\qquad \theta p_\theta(\tau, \theta) = e_\tau(\tau, \theta) + p(\tau, \theta) = \frac{1}{\theta}$

so that $p_\theta = 1/\theta^2$, and (1.10) holds. The equation of state for p is now determined:

(3.3) $\qquad p(\tau, \theta) = p_m(\tau) + (1/m(\tau) - 1/\theta)$

Our construction will be complete once $e_\theta(\tau, \theta)$ is determined. For then, since p_0, s, and $m(\tau)$ determine p, $e_\tau(\tau, \theta)$ is known from (3.2). Then e_τ and e_θ determine $e(\tau, \theta)$, and $S(\tau, \theta)$ is determined from the Gibbs relation (1.2). ((3.2) ensures that the expression obtained for dS can be integrated.)

We must have $(e_\theta)_\tau = (e_\tau)_\theta = -2/\theta^2$, so e_θ has the form

$$e_\theta(\tau, \theta) = e_\theta(\tau_0, \theta) - 2(\tau - \tau_0)/\theta^2$$

We specify $e_\theta(\tau_0, \theta)$ so that (1.8) holds in the region of interest. It suffices to define

$$e_\theta(\tau_0, \theta) = 80\tau_0(\theta^{-2} + \theta_0^{-2})$$

so that $e_\theta(\tau, \theta) > 0$ for $\theta > 0, 0 < \tau < 40\tau_0$. An explicit formula for $e(\tau, \theta)$ is

(3.4)
$$e(\tau, \theta) = 80\tau_0(\theta_0^{-2} - \theta^{-2})\theta - \int_{\tau_0}^{\tau} 1/m(\tau)\, d\tau$$
$$+ (\tau - \tau_0)(2/\theta - p_0 + \tfrac{1}{2}s^2(\tau - \tau_0)) + e_0$$

We must now show that $m(\tau)$, p_0 and s may be specified so that Fig. 3 is

valid and (1.9) holds, along with (1.11), (1.14), (1.15). Observe that $-sM(\tau,\theta)$ is independent of p_0 and s:

$$-sM(\tau,\theta) = p(\tau,\theta) - p_m(\tau) = 1/m(\tau) - 1/\theta$$

Using (2.7), we must have

$$\frac{1}{-s}L_\tau(\tau,\theta) = \theta p_\theta - (-s)M = 2/\theta - 1/m(\tau)$$

Requiring $L(\tau_0,\theta_0) = 0$, the quantity $(1/-s)L(\tau,\theta)$ is independent of p_0 and s

(recall $(1/-s)L_\theta = e_\theta$):

$$(3.5) \qquad \frac{1}{-s}L(\tau,\theta) = 80\tau_0(\theta_0^{-2} - \theta^{-2})\theta + 2(\tau - \tau_0) - \int_{\tau_0}^\tau 1/m(\tau)\,d\tau$$

We will use this formula to manipulate the curve $(L = 0)$ by choice of $m(\tau)$. Set $\tau_k = (k+1)\tau_0$. We shall show that $m(\tau)$ can be chosen so that:

$$\begin{aligned}
&\text{i)} \quad L(\tau,m(\tau)) > 0 \quad \text{for} \quad \tau - \tau_0 > 0 \quad \text{small}\\[4pt]
(3.6) \quad &\text{ii)} \quad L(\tau,\theta_0+\varepsilon_1) < 0 \quad \text{for} \quad \tau_3 < \tau < \tau_5 \quad \text{for some } \varepsilon_1 > 0 ,\\
&\qquad\quad \text{while} \quad \theta_0 < m(\tau) < \theta_0+\varepsilon_1 \quad \text{for} \quad \tau_4 < \tau < \tau_5\\[4pt]
&\text{iii)} \quad L(\tau,m(\tau)) > 0 \quad \text{for} \quad \tau \quad \text{large}
\end{aligned}$$

We ask that $m(\tau)$ be defined and smooth for $\frac{1}{2}\tau_0 < \tau < 40\tau_0$, and satisfy conditions (m1)-(m3) below:

(m1) $\qquad m(\tau_0) = \theta_0 \quad$ and $\quad m'(\tau_0) = 0$

Since $(1/-s)L_\tau(\tau_0,\theta_0) = 1/\theta_0 > 0$, (3.6i) follows.

$$\begin{aligned}
\text{(m2)} \qquad & m'(\tau) < 0 \qquad \text{for} \quad \tau_0 < \tau < \tau_1 ,\\
& m(\tau) = \theta_0/6 \quad \text{for} \quad \tau_1 < \tau < \tau_3 ,\\
& m'(\tau) > 0 \qquad \text{for} \quad \tau_3 < \tau < \tau_4 , \quad \text{and}\\
& \max_{\tau_4 < \tau < \tau_5} m(\tau) = \theta_0+\varepsilon_1 , \quad \varepsilon_1 > 0 \text{ small, with} \quad m(\tau_4) = \theta_0 = m(\tau_5) .
\end{aligned}$$

To verify (3.6ii), we estimate, for $\tau_3 < \tau < \tau_5$,

$$\begin{aligned}
\frac{1}{-s}L(\tau,\theta_0+\varepsilon_1) &< C\varepsilon_1 + 2(\tau_5 - \tau_0)/\theta_0 - (\tau_3 - \tau_1)6/\theta_0\\
&= C\varepsilon_1 - 2\tau_0/\theta_0 < 0 \quad \text{if } \varepsilon_1 \text{ is small.}
\end{aligned}$$

(m3) $m(\tau) = \theta_0$ for $\tau > \tau_5$

To verify (3.6iii), we estimate, for $\tau > \tau_6$,

$$\frac{1}{-s}L(\tau,\theta_0) > 2(\tau - \tau_0)/\theta_0 - ((\tau_4-\tau_0)6/\theta_0 + (\tau - \tau_4)/\theta_0)$$

$$= ((\tau - \tau_4) - 4(\tau_4 - \tau_0))/\theta_0 > 0 \text{ if } \tau > \tau_{20} .$$

Now: Since $(1/-s)L_\theta(\tau,\theta) > c > 0$ for $\frac{1}{2}\tau_0 < \tau < \tau_{40}$, the set $(L = 0)$ is a curve $\theta = \ell(\tau)$ which is bounded above. We claim that $\ell(\tau)$ is defined for $\tau_0 < \tau < \tau_5$. Indeed, $L(\tau,\theta_0) < 0$ for $\tau_3 < \tau < \tau_5$, and since $L_\tau(\tau,\theta_0/6) > 0$ for $\tau < \tau_3$ (use (2.7) with $M < 0$), we have $L(\tau,\theta_0/6) < 0$ for $\tau < \tau_3$, so $\ell(\tau)$ is bounded below. In fact, we have shown that $\ell(\tau) > m(\tau)$ for $\tau_1 < \tau < \tau_5$.

We now set

$$\tau_+ = \min\{ \tau \mid \tau > \tau_5 \text{ and } \ell(\tau) = m(\tau) \}$$

$$\tau_- = \max\{ \tau \mid \tau < \tau_1 \text{ and } \ell(\tau) = m(\tau) \}$$

Then $\tau_5 < \tau_+ < \tau_{20}$, and since $m'(\tau_0) = 0 > \ell'(\tau_0)$, $\tau_0 < \tau_- < \tau_1$. We set $\theta_+ = m(\tau_+)$, $\theta_- = m(\tau_-)$, fixing Z_+ and Z_-, and completing the construction of Fig. 3.

It remains to choose p_0 and s so that $p(\tau,\theta) > 0$ and (1.9), (1.14), (1.15) hold in a suitable domain. But

(3.7) $p_\tau(\tau,S) = p_\tau(\tau,\theta) - \theta p_\theta(\tau,\theta)^2/e_\theta(\tau,\theta)$

(One calculates $p(\tau,S(\tau,\theta))_\tau$ and $\theta_\tau(\tau,S) = -S_\tau/S_\theta(\tau,\theta)$ and uses (2.1),(2.2).) So (1.15) implies (1.9). But from (3.2),

(3.8) $p_\tau(\tau,\theta) = -s^2 + (1/m(\tau))'$

So we can ensure that $p_\tau(\tau,\theta) < 0$ for $0 < \tau < \tau_{40}$ if we choose s^2 sufficiently large.

As we choose p_0, we seek to fix the domain of states Ω_p in the (τ,p) plane so that Ω_p is convex, $p > 0$ in Ω_p, (1.14) holds, and, pulled back to the (τ,θ) plane, the domain Ω_θ contains (say) the half strip

$\{(\tau,\theta)|\ 0 < \tau < \tau_{40}\ ,\ \theta > \theta_0/10\ \}$ which contains all our constructions. It suffices to choose Ω_p of the form

$$\Omega_p\ =\ \{(\tau,p)|\ 0 < \tau < \tau_{40}\ \text{and}\ p - p_0 + s^2(\tau - \tau_0) > -10/\theta_0\ \}$$

with

(3.9) $$p_0\ =\ 40\tau_0 s^2 + 10/\theta_0$$

To check (1.14), compute

$$e_\tau(\tau,p)\ =\ e_\tau(\tau,\theta) + e_\theta(\tau,\theta)(-p_\tau(\tau,\theta)/p_\theta(\tau,\theta))$$

Using (3.2) and (3.8) one may verify that $e_\tau(\tau,p) > 0$ in Ω_p if s^2 is sufficiently large.

Now the equation of state is completely specified, and (1.8)-(1.15) hold except for (1.12). We have already noted that the entropy condition s(E) holds, since $\ell(\tau) > m(\tau)$ for $\tau_- < \tau < \tau_+$.

The penultimate stage of our analysis is to show that for λ/μ sufficiently large, there is a trajectory of (1.6) leaving Z_0 and intersecting the curve (M = 0) before the "hump". Since $m'(\tau_0) > \ell'(\tau_0)$, Z_0 is a saddle point for (1.6) (see (2.9)). If κ is the positive characteristic value for (1.6) at Z_0, the the characteristic vector $(\kappa - \tau L_\theta/\lambda, \tau L_\tau/\lambda)$ may be shown to have positive components. Let $Z(\xi)$ be the solution leaving Z_0 along this vector. So long as $Z(\xi)$ remains above the curve (M = 0), τ is increasing on $Z(\xi)$ and we may write $\theta = z(\tau)$ along $Z(\xi)$. Then

$$\frac{dz}{d\tau}\ =\ \frac{\mu L(\tau,z(\tau))}{\lambda M(\tau,z(\tau))}$$

It is clear that $z(\tau)$ is defined and increasing for $\tau_0 < \tau < \tau_*$, where

$$\tau_*\ =\ \min\{\ \tau\ |\ \tau > \tau_0\ \text{and}\ L(\tau,\theta_0) = 0\ \}$$

Note that $\tau_* < \tau_3$. We will "channel" the curve $z(\tau)$ under the curve (L = 0) by using a thin box with a corner cut out: Define

$$R^*\ =\ (\tau_0,\tau_3) \times (\theta_0 - \varepsilon_2, \theta_0 + \varepsilon_1)\ \setminus\ (\tau_0,\tau_*) \times (\theta_0 - \varepsilon_2, \theta_0 + \tfrac{1}{2}\varepsilon_1)$$

Here $\varepsilon_2 > 0$ is chosen so that $\bar{R}^*$ is contained in the region (M > 0). Now, the

trajectory $Z(\xi)$ must enter R^* on the cut-out part of the boundary with

$\theta > \theta_0$, i.e., it must enter at a point $(\tau, \theta_0 + \frac{1}{2}\varepsilon_1)$ for $\tau_0 < \tau < \tau_*$, or at a

point (τ_*, θ) for $\theta_0 < \theta < \theta_0 + \frac{1}{2}\varepsilon_1$.

Now $\sup\limits_{R^*} |L/M|$ is finite. If λ/μ is sufficiently large in R^*, we may

obtain

$$\left| \frac{dz}{d\tau} \right| < \min\{ \tfrac{1}{2}\varepsilon_1/(\tau_3 - \tau_0), \ \varepsilon_2/(\tau_3 - \tau_*) \}$$

so long as $Z(\xi)$ is in R^*. This implies that the trajectory $Z(\xi)$ must leave R^*

on its right boundary, $\{\tau_3\} \times (\theta_0 - \varepsilon_2, \theta_0 + \varepsilon_1)$, which is contained in the region

($L < 0$ and $M > 0$). Past this point, θ decreases and τ increases along $Z(\xi)$,

so since $m(\tau)$ acheives the value $\theta_0 + \varepsilon_1$, $Z(\xi)$ must intersect the curve ($M = 0$)

at a point $(\tilde{\tau}, \tilde{\theta})$, where $\tau_5 > \tilde{\tau} > \tau_3 > \tau_-$.

We may now easily construct a positively invariant region containing Z_-

and disjoint from Z_+, so that no trajectory can connect Z_- to Z_+. The region

is bounded on the right by the line $\tau = \tilde{\tau}$ for $\theta < \tilde{\theta}$ ($M < 0$ here), from above

by the curve $\theta = z(\tau)$ for $\tau_0 < \tau < \tilde{\tau}$ (trajectories cannot cross), and by the

curve $\theta = \min\{\ell(\tau), m(\tau)\}$ for $\tau < \tau_0$ (the vector field either points down, or

to the left at a point where $\ell'(\tau) < 0$, as in §2). Our example is complete.

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*
    A classical result of Gilbarg states that a simple shock wave solution of Euler's equations is compressive if and only if a corresponding shock layer solution of the Navier-Stokes equations exists, assuming, among other things, that the equation of state is convex. An "entropy condition" appropriate for weeding out "unphysical" shocks in the nonconvex case has been introduced by T.-P. Liu. For shocks satisfying his entropy condition. Liu showed that purely viscous shock layers exist (with zero heat conduction). Dropping the convexity assumption, but retaining many other reasonable restrictions on the equation

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ABSTRACT (continued)

of state, we construct an example of a (large amplitude) shock which satisfies
Liu's entropy condition but for which a shock layer does <u>not</u> exist if heat
conduction dominates viscosity. We also give a simple restriction, weaker than
convexity, which does guarantee that shocks which satisfy Liu's entropy condition
always admit shock layers.

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